Resit Kaleidoscope Mathematics (WPMA18002)

Statistics and Probability

Monday December 3, 19:00-22:00, A. Jacobshal 01

- During the exam it is allowed to use a simple calculator such as the CASIO FX-82 or TI30.
- Give in all cases the reasoning leading to your answer.
- Provide each page with your name and student number.
- Use a sufficient number of decimals to make your computations unambiguously.
- This test consists of 4 exercises. In total you can score 100 points for it. We wish you lots of success with its completion!

1. **Deck of Cards**. Each of four decks of 52 cards are carefully shuffled such that their order is completely random. Suppose you receive one card from each deck. Let X be the number of aces received. Compute the probability mass function of X. $\boxed{20}$

2. **Each Player One Ace**. A deck of 52 cards is distributed completely ad random among four players such that each player gets 13 cards. Find the probability that each player gets exactly one ace. $\boxed{20}$

3. **Reading errors**. The alignment of the magnetic tape and head in a storage system affects the performance of the system. Suppose that in 10% of the read operations there are skewed alignments, 5% of the read operations there are off-center alignments, and the remaining read operations are properly aligned. The probability of a read error is 0.01 from a skewed alignment, 0.02 from an off-center alignment, and 0.001 from a proper alignment.

   (a) $\boxed{15}$ What is the probability of a read error?

   (b) $\boxed{15}$ If a read error occurs, what is the probability that it is due to a skewed alignment? And, what is the probability of a read error due to off-center alignment? Finally, what is probability of a read error in case of proper alignment? What is your conclusion?

see next page

4. Consider a Markov chain $\{S_n\}_{n\geq 1}$ with states $\{1, 2\}$ and probability transition matrix

$$P = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

(a) $\boxed{5}$ Is the chain irreducible?

(b) $\boxed{5}$ What is the proper interpretation of the event $S_{n+2} = 2 | S_n = 1$.

(c) $\boxed{10}$ Compute the probability $P(S_{n+2} = 2 | S_n = 2)$.

(d) $\boxed{10}$ Find the stationary distribution of the chain.

Modular Arithmetic

- Give in all cases the reasoning leading to your answer.
- Provide each page with your name and student number.
- This part of the test consists of 3 exercises. In total you can score 10 points for it, one point you get for free.

1. (a) [1 point.] Determine integers x, y such that $x \cdot 10^2 + y \cdot 13^2 = 42$.

   (b) [2 points.] Suppose that x, y, a, b are integers and that $x \cdot a^2 + y \cdot b^2 = 42$. Show that $\gcd(a, b) = 1$.

2. Denoting elements a mod 100 by $\overline{a}$, and with x considered as an element of $\mathbb{Z}/100\mathbb{Z}$,

   (a) [1 point] Find an x such that $\overline{17} \cdot x + \overline{5} = \overline{24}$.

   (b) [1 point] Does x exist such that $\overline{36} \cdot x = \overline{10}$?

   (c) [1 point] How many of the elements of $\mathbb{Z}/100\mathbb{Z}$ are units?

3. (a) [2 points.] Prove that $5 | (2^n + 3)$ if and only if $4 | (n - 1)$.

   (b) [1 point.] Show that $2^{10} \equiv -1 \bmod 25$ and that $25 | (2^{81} + 3)$.

Dimensional Analysis

- All answers need to be justified.
- Each exercise of this part has a certain amount of points, summing up to 9 points. The grade of this part will be computed as $grade = 1 + (points\ obtained)$. Please answer each exercise on a different sheet to facilitate the grading.
- A table with dimensions can be found on the last page of this exam.
- Please be aware that, in the context of this course, squared brackets are used for denoting dimensions of a physical quantity. If used otherwise, results will be considered wrong.

1. In cardiology, the pressure difference δ across a narrowed blood vessel is usually assumed dependent on the following variables: the mass density of the blood ρ, the average velocity before the valve v_1, the average velocity after the valve v_2, the area that the fluid crosses before the valve S_1, and the area S_2 after the valve.

   (a) $\boxed{2.0}$ Assume that δ depends only on v_1, v_2, ρ. Using dimensional analysis, write a mathematical model for δ. Indicate explicitly the non-dimensional groups, if any appear.

   (b) $\boxed{1.0}$ Explain if the mathematical model that you obtained in Exercise 1(a) could take or not the particular form of the so-called *Bernoulli equation*?

   $$\delta = \frac{\rho}{2}v_2^2 - \frac{\rho}{2}v_1^2$$

   (c) $\boxed{2.0}$ Using dimensional analysis, build a mathematical model for δ in terms of ρ, v_1, v_2, S_1, S_2. Verify that if you remove the dependence on S_2, you obtain the same model than in Exercise 1(a).